Convex Optimization Problem I

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배울 것들

- Line, Line segment
- Affine set, Affine combination, Affine hull

Line and line segment Let x_1 and x_2 be point in $\mathbb{R}^n$. Let $y = \theta x_1 + (1-\theta)x_2 = x_2 + \theta(x_1 - x_2)$ for $\theta \in \mathbb{R}$. y is a point on the line passing through x_1 and x_2.

For $\theta \in [0, 1]$ y is a line segment.

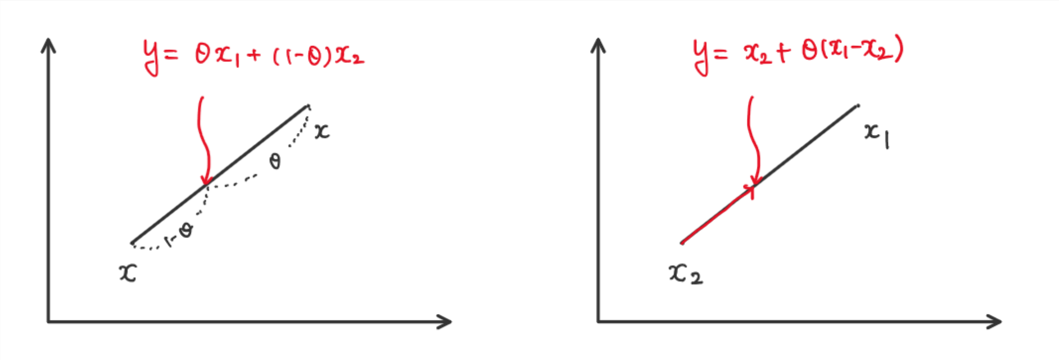

Figure 1: Illustration of a line segment

Affine set

A set $C \subset \mathbb{R}^n$ is affine if any $x_1, x_2 \in C$ and $\theta \in \mathbb{R}$, we have $\theta x_1 + (1-\theta)x_2 \in C$.

Affine combination of points

Let $x_1, \cdots, x_m \in \mathbb{R}^n$. $y = \sum_{j=1}^m \theta_j x_j$ for $\sum_{j=1}^m \theta = 1$. Then y is called an affine combination of $x_1, \cdots, x_m$.

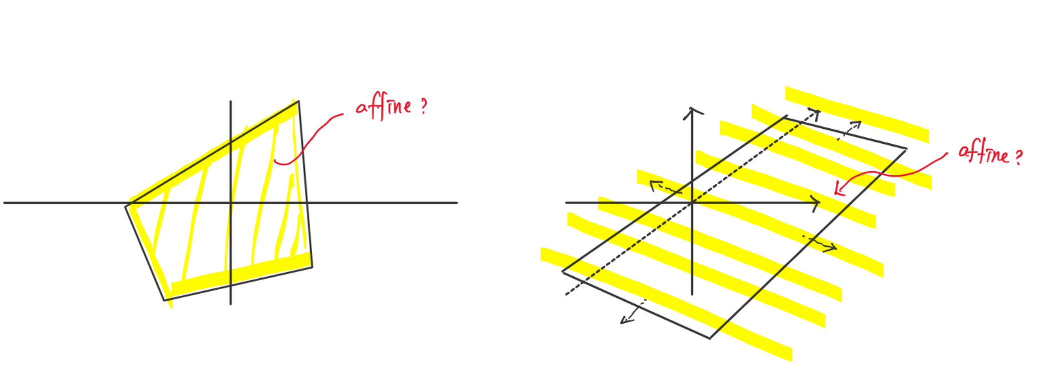

Figure 2: Illustration of an affine set

An affine set is characterized as a linear subspace plus an offset.

- C is an affine set and $x_0 \in C$, then $V = C - \{x_0\}$ is a linear subspace. That is, $C = \{x_0\} + V$.
- Conversely, if V is a linear space, then $C = V + \{x_0\}$ is an affine set.

proof) see Appendix A.

Hyperplane is an affine set.

$$C = \{x \in \mathbb{R}^n : a^\top x = b\}$$

Suppose that C is not empty. Let $a^\top x_0 = b$, then $C = \{x : a^\top(x - x_0) = 0\} = \underbrace{\{x : a^\top x = 0\}}_{\text{subspace}} + \{x_0\}$

.

See the following statement with considering the definition of affine set.

- Let $\beta_0 \in \mathbb{R}^n$, $\boldsymbol{\beta} \in \mathbb{R}^p$ and X be $n \times p$ matrix. Let $y = \beta_0 + X\boldsymbol{\beta}$, then the collection of y for all $\boldsymbol{\beta}$ consists of an affine space in $\mathbb{R}^n$.
- Let $C = \{x \in \mathbb{R}^p : Ax = b\}$. If C is not empty, then C is an affine set. Let $x^* \in C$ then $Ax^* = b$. Let null$(A) = \{x \in \mathbb{R}^p : Ax = 0\}$ then, null$(A) + x^* \subset C$. Conversely, $C - \{x^*\} \subset$ null(A). That is, $C =$ null$(A) + \{x^*\}$ and clearly null(A) is subspace. So, C is affine set.

Affine hull

$$\text{aff}(C) = \{\theta x_1 + \cdots + \theta_k x_k : x_1, \cdots, x_k \in C, \theta_1 + \cdots + \theta_k = 1\}$$

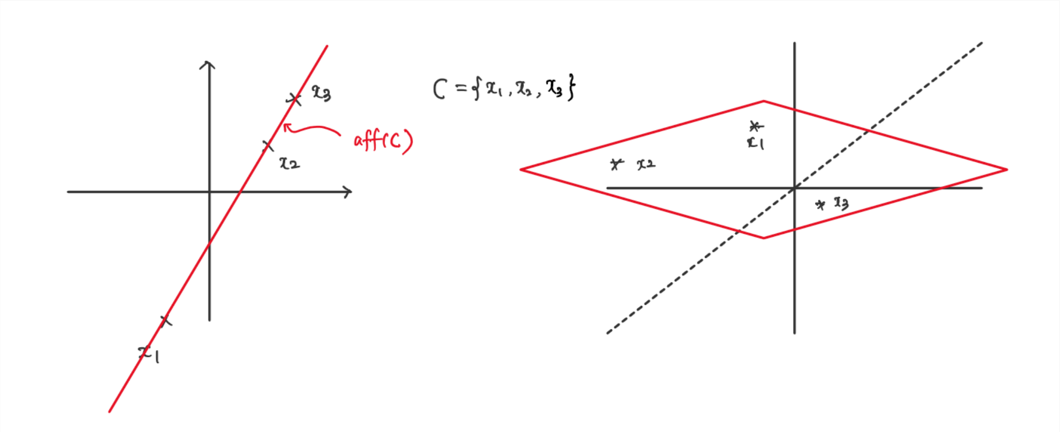

Figure 3: Illustration of an affine set

Affine combination

Let $x_1, \cdots, x_k \in \mathbb{R}^n$. For $\sum_{j=1}^k \theta_j = 1$, $y = \sum_{j=1}^k \theta_j x_j$ is called an affine combination of $x_1, \cdots, x_k$.

Proposition Affine hull is an affine set.

We can prove that any set of all affine combinations of $x_1, \cdots, x_k$ is an affine set:

$$C = \{y : y = \sum_{j=1}^{k} \theta_j x_j, \text{ where} \sum_{j=1}^{k} \theta_j = 1\}$$

is affine set.

Proof) See Appendix B

Characterization of affine hull

aff(C) is the smallest affine set containing C, in the sense: if S is any affine set with $C \subset S$ then aff$(C) \subset S$.

proof) See Appendix C.

Affine dimension

Relative interior*: relative interior in low dimension

Consider a set $C = \{(x_1, x_2, x_3)^\top \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 1, x_3 = 0\}$. Where is an interior in $\mathbb{R}^3$? For all $x \in C$, there is no ball $B(x, r) \subset C$, implying no interior point in C. So, we introduce a new definition, a relative interior.

$$\text{relint } C = \{x \in C : B(x, r) \cap \text{aff } C \subset C \text{ for some } r > 0\}.$$

What is the origin of the name ‘relative interior’?

Let (X, τ) be topological space, where τ is a family of all open sets of X. Consider $A \subset X$ then $\{o \cap A : o \in \tau\}$ is also a topology, called the relative topology for the subset A. (A need not be open)

Since $B(x, r)$ is open, the definition indicates the way to construct the relative topology for the subset aff(C).

- 어떤 set A의 interior, exterior, boundary 구분 방법은 다음과 같다.
  - 어떤 점 $x \in A$에서 중심이 x 반지름이 r인 작은 Ball $B(x,r)$을 잡아서 $B(x,r) \subset A$ 이 가능하면, x는 interior of A.
  - 어떤 점 $x \notin A$에서 중심이 x 반지름이 r인 작은 Ball $B(x,r)$을 잡아서 $B(x,r) \subset A^c$ 이 가능하면, x는 exterior of A.
  - 어떤 점에서 임의로 $B(x,r)$을 잡아도 항상 $A \cap B(x,r) \neq \phi$, 그리고 $A^c \cap B(x,r) \neq \phi$면 그 점은 boundary of A.

Appendix

Appendix A

(Definition of subspace)

Let V be vector space. We call S the subspace of V if

- $S \subset V$
- $0 \in S$
- If $x, y \in S$, $ax + by \in S$ for scalar a and b.

Example) column space, null space...

Appendix A

(Proof) Let C be an affine set and let $v_1, v_2 \in C$. It suffices to prove that V is subspace: $0 \in V$ and

$$\alpha(v_1 - x_0) + \beta(v_2 - x_0) \in V$$

for $\alpha, \beta \in \mathbb{R}$. $0 \in V$ is trivial since $x_0 \in C$. In addition, it is clear that

$$\alpha(v_1 - x_0) + \beta(v_2 - x_0) = \underbrace{\alpha v_1 + \beta v_2 + (1 - \alpha - \beta)x_0}_{\in C \text{ by proposition 1}} - x_0 \in V,$$

which completes the proof.

Appendix B

Proposition 1

Any affine combination of points in an affine set is included in the affine set again.

(Proof) Let S be an affine set and choose $y_1, \cdots, y_m \in C$. It suffices to prove that $\sum_{j=1}^{m} \alpha_j y_j \in C$ for $\sum_{j=1}^{m} \alpha_j = 1$. Let $y_1, y_2 \in S$ and $y_1 \neq y_2$, then $\beta_1 y_1 + \beta_2 y_2 \in C$ for $\beta_1 + \beta_2 = 1$ by definition. Suppose that $\beta_1 y_1 + \cdots + \beta_{k-1} y_{k-1} \in C$ for $\sum_{j=1}^{k-1} \beta_j = 1$ and $y_k \in C$. Since C is affine set,

$$t(\beta_1 y_1 + \cdots + \beta_{k-1} y_{k-1}) + (1-t) y_k \in C.$$

By letting $\alpha_1 = t\beta_1, \cdots, \alpha_{k-1} = t\beta_{k-1}$ and $\alpha_k = (1-t)$, we know that $\sum \alpha_j = 1$. Mathematical induction completes the proof.

Appendix B

(proof) Choose arbitrary $y, y' \in C$ and let $y = \sum_{j=1}^{k} \theta_j x_j$ and $y' = \sum_{j=1}^{k} \theta'_j x_j$ where $\sum_{j=1}^{k} \theta_j = \sum_{j=1}^{k} \theta'_j = 1$. Then, for $\alpha \in \mathbb{R}$

$$
\begin{aligned}
\alpha y + (1-\alpha) y' &= \sum_{j=1}^{k} \alpha \theta_j x_j + \sum_{j=1}^{k} (1-\alpha) \theta'_j x_j \\
&= \sum_{j=1}^{k} (\alpha \theta_j + (1-\alpha)\theta'_j) x_j .
\end{aligned}
$$

$\sum_{j=1}^{k} (\alpha \theta_k + (1-\alpha)\theta'_k) = \alpha \sum_{j=1}^{k} \theta_k + (1-\alpha) \sum_{j=1}^{k} \theta'_k = \alpha + (1-\alpha) = 1$, which completes the proof.

Appendix C

Proposition 1 을 이용하면 다음 결론에 이른다.

- 임의의 몇 개 점을 이용해 Affine hull을 만들면 집합은 Affine set이 된다.
- 어떤 Affine set이 주어진 경우 그 집합의 임의원소로 Affine hull을 만들어도 새로 만들어진 원소는 원래 Affine set이 모두 가지고 있다.

Let C_0 be an affine set containing $y_1, \cdots, y_m$ and $\text{aff}(C_0)$ be affine hull induced by C_0. Then, $\text{aff}(C_0) \subset C_0$. Since C_0 is the affine set containing $y_1, \cdots, y_m$, we know that $\text{aff}(C_0)$ is the smallest affine set containing $\{y_1, \cdots, y_m\}$.