Linear algebra for computational statistics III

Jong-June Jeon

September, 2025

Department of Statistics, University of Seoul

Things to know

- basic operation of matrix
- spanning space, null space
- projection and geometry
- linear map and matrix

Decomposition of matrix

Decomposition of linear maps

Step 3

행렬이 대응시키는 변환을 분해하는 과정을 소개한다. 먼저 내적을 도입하여, 벡터공간 위에서 거리와 각도가 자연스럽게 정의되는 과정을 살펴본다. 다음으로 대칭인 반양정치행렬의 분해를 특별한 직교 선형변환의 분해로 이해할 수 있으며, 이를 통해 행렬의 대응을 분해하여 해석한다. 여기서는 내적공간(inner product space)와 정사형(projection), Spectral Decomposition, Singular Value Decomposition을 배운다.

Inner product An inner product space is a vector space V with an inner product:

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

that satisfies the following three properties for all vectors $x, y, z \in V$ and all scalars $a \in \mathbb{R}$.

- Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
- Linearity:

$$\begin{aligned} \langle ax, y \rangle &= a\langle x, y \rangle \\ \langle x + y, z \rangle &= \langle x, z \rangle + \langle y, z \rangle \end{aligned}$$

- Positive-definite: $\langle x, x \rangle > 0, x \in V - \{0\}$

example

Suppose that $a, b \in \mathbb{R}^p$

- Let $\langle a, b \rangle = a^\top b$. Then the $\langle \cdot, \cdot \rangle$ is inner product?
- Let $H \in \mathbb{R}^{p \times p}$ is symmetric and $\langle a, b \rangle = a^\top Hb$. Then the $\langle \cdot, \cdot \rangle$ is inner product?
- If H is positive definite, ...

(NOTE) Vector space에는 Addition과 scalar multiplication 연산만 정의되어 있다. Vector space 위에 inner production 연산을 정의해놓으면, Vector space 위에 각도를 정의할 수 있다. 한편 inner production 연산이 주어지면 원소의 길이(norm) 혹은 두 원소간의 거리 (distance)를 정의할 수 있다.

- For $x, y \in V$ define $\langle x, y \rangle = x^\top y$. If $x^\top y = 0$ we write $x \perp y$
- We define the norm of $x \in V$ by $\|x\| = \sqrt{x^\top x}$
- We can define the distance between x and y by $d(x, y) = \|x - y\|$

Hereafter, we use the above definition of the inner product and the norm in our vector space V.

angle and inner product (law of cosine)

- Let a point A, B, C in $\mathbb{R}^2$ and C is the origin and B is a point on x-axis.
- Let the length of $\overline{\mathrm{AB}}$, $\overline{\mathrm{BC}}$ and $\overline{\mathrm{CA}}$ be c, a, and b, respectively.
- Let the angle $\angle C$ be θ.
- The point A is $(b\cos\theta, b\sin\theta)$, and the point $B = (a, 0)$. Thus,

$$\begin{aligned} c^2 &= (b\cos\theta - a)^2 + b^2\sin^2\theta \\ &= a^2 + b^2 - 2ab\cos\theta \end{aligned}$$

angle and inner product

Because the law of cosine is the fact derived only from geometry, we can apply the law of cosine to a Euclidean space.

Consider a vector u, v, and $u-v$ and denote the norms of the vectors by $\|u\|$, $\|v\|$, and $\|u-v\|$, respectively. Note that $\|u\|$, $\|v\|$, and $\|u-v\|$ correspond to b, a, and c. By the law of cosine

$$\|u-v\|^2 = \|u\|^2 + \|v\|^2 - 2\|u\|\|v\|\cos(\theta),$$

which reduces to

$$u^\top v = \|u\|\|v\|\cos(\theta).$$

As a result the angle in $\mathbb{R}^p$ are defined by the law of cosine.

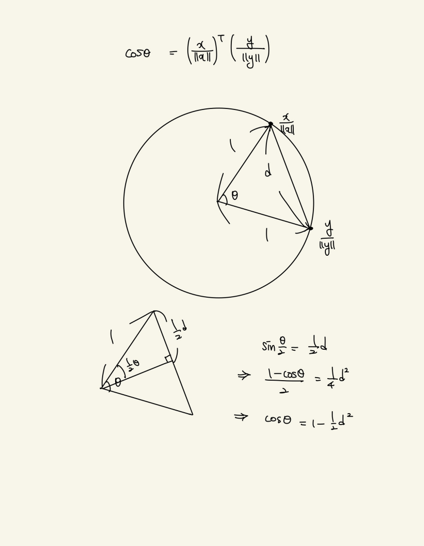
angle and inner product Thus,

- $\cos(\theta) = u^\top v / (\|u\| \|v\|)$
- $u^\top v = 0$ is regarded as $u \perp v$

ch) Let u and v be points on a unit sphere and let d be a Euclidean distance between u and v. Then,

$$u^\top v = \cos(\theta) = 1 - \frac{1}{2} d^2.$$

The equation shows the relationships of inner product, cosine similarity, and distance.


cosθ = (x/‖x‖)^T (y/‖y‖)
x/‖x‖
y/‖y‖
1
1
d
θ
1
1/2 d
1/2 θ
θ
sin θ/2 = 1/2 d
⇒ (1−cosθ)/2 = 1/4 d²
⇒ cosθ = 1 − 1/2 d²

Projection

Suppose that V is an inner product vector space. Let $x, y \in V$ then there exists $\hat{y} \in \mathcal{C}(x)$ such that $(y - \hat{y}) \perp x$. That is y is decomposed into $y = \hat{y} + (y - \hat{y})$ with $\hat{y} \perp (y - \hat{y})$.

Transpose and projection (figure will be corrected!)

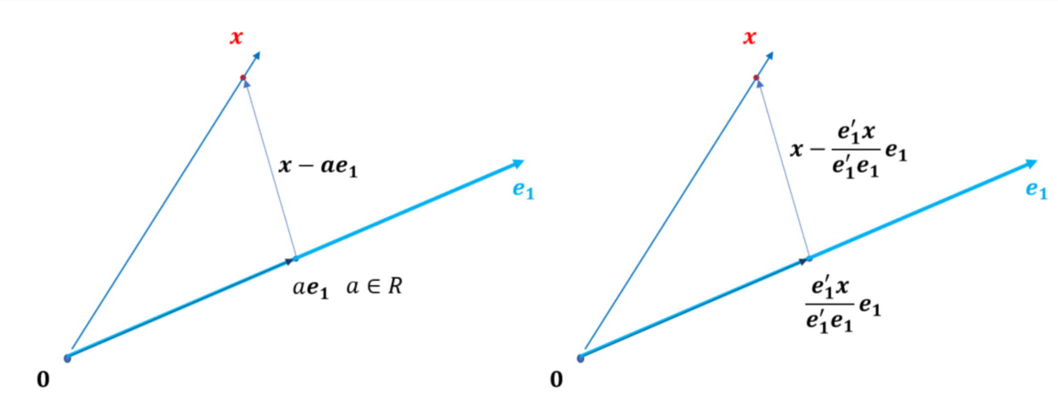


Figure 1: Illustration of projection via transpose operation

$y = ax + (y - ax)$ with $a = y^\top x / \|x\|^2 \in \mathbb{R}$

Projection Let $y, x_1, \cdots, x_k \in V$ and suppose that $x_1, \cdots, x_k$ are linearly independent. Consider $X = [x_1, \cdots, x_k]$ and $\mathcal{C}(X)$. Then, how can we find $\hat{y} \in \mathcal{C}(X)$ such that

$$y = \hat{y} + (y - \hat{y})$$

satisfying $\hat{y} \perp (y - \hat{y})$?

The answer is the Projection map (matrix)!

Orthogonal Projection Let $x, y \in \mathbb{R}^p$ and consider Π, a linear map from $\mathbb{R}^p$ to $\mathbb{R}^p$. The projection map of y onto $\mathcal{C}(x)$ satisfies the following properties:

- $\Pi y \in \mathcal{C}(x)$;
- $(\Pi y)^\top y = y^\top (\Pi y) = 0$;
- $\Pi(\Pi y) = \Pi y$

Orthogonal Projection

Orthogonal projection is a linear map defined by $\Pi : V \mapsto V$ (in fact, we can understand the P as a matrix) such that $\Pi = \Pi^2 = \Pi^\top$.

- Let $\Pi = x(x^\top x)^{-1}x^\top$ then $\Pi = \Pi^2 = \Pi^\top$?
- $\Pi y \in \mathcal{C}(x)$ for $x \in V$?
- $\langle \Pi y, y - \Pi y \rangle = 0$?

You can conclude that the Π is the orthogonal projection operator (onto $\mathcal{C}(x)$).

Orthogonal Projection

- Let $x, y \in \mathbb{R}^n$ and compute the projection of y onto $\mathcal{C}(x)$. Then, it is given by $(x(x^\top x)^{-1}x^\top)y$. Show that $x(x^\top x)^{-1}x^\top$ is a projection operator.
- Let $X \in \mathbb{R}^{n\times m}$ and $y \in \mathbb{R}^n$. Compute the projection of y onto $\mathcal{C}(X)$.
- Write an example and confirm the result numerically.

잠깐!

회귀모형의 OLS를 돌이켜보자. Response vector $Y \in \mathbb{R}^n$과 predictor matrix $X \in \mathbb{R}^{n\times p}$ 일때, OLS를 이용한 Y값의 추정량은

$$\hat{Y} = X(X^\top X)^{-1}X^\top Y \in \mathbb{R}$$

로 주어진다. 여기서 $X(X^\top X)^{-1}X^\top$가 $\mathcal{C}(X)$에 Projection operator 고 $\hat{Y} \in\in \mathcal{C}(X)$ 이며 $(Y - \hat{Y}) \perp \hat{Y}$임을 알 수 있다.

Projection은 벡터 성분을 직교분해할 때 흔히 볼 수 있었던 연산이다.

Basic on undergraduate levels

- Symmetric Matrix
- Orthogonal Matrix

Orthogonal matrix

- Orthogonal matrix E: a square matrix satisfying

$$E = [e_1, \cdots, e_p],$$

  where $e_j^\top e_k = 0$ for $j \neq k$ and $\|e_j\| = 1$ for all j.
- It is easily shown that $E^\top E = I$.
- Because $E(E^\top E) = E$, $EE^\top = I$.

That is, $E^\top E = EE^\top = I$, and $E^\top$ is the inverse of E.

Orthogonal matrix (Isometric transformation)

Let $E \in \mathbb{R}^{p \times p}$ be orthogonal matrix and $x, y \in \mathbb{R}^p$. $d(x,y) = d(Ex, Ey)$?

$$\begin{aligned} d(x,y)^2 &= (x-y)^\top (x-y) = (x-y)^\top E^\top E(x-y) \\ &= \|E(x-y)\|^2 = d(Ex, Ey)^2 \end{aligned}$$

The map $\mathcal{L}_E$ preserves the distance (isometric). Actually, E is understood as the rotation map.

Orthogonal matrix (Rotation)

What is the geometrical meaning of the first column of an orthogonal matrix?

Let E be the orthogonal and $a_1 = (1, 0, \cdots, 0)$.

- Ea_1 is the first column vector.
- In addition, Ea_1 is the image of a linear map E for a_1, which is the first coordinate basis vector.
- That is, the first column vector is the transformed image of the first coordinate basis vector.

To sum up, each column of E denotes an image of each coordinate basis transformed by E. Since the transformed image is orthogonal to each other, the map can be regarded as a geometrical rotation.

Diagonal matrix

Let $D = diag(d_1, \cdots, d_p) \in \mathbb{R}^{p \times p}$ be diagonal matrix and $x = (x_1, \cdots, x_p) \in \mathbb{R}^p$. Then,

$$Dx = (d_1 x_1, \cdots, d_p x_p)^\top$$

The map D is called the scaling map.

앞다행, 뒤다열!

Eigendecomposition

Let $A \in \mathbb{R}^{p \times p}$ be a symmetric matrix. Then there exists an orthogonal matrix E and a diagonal matrix D (with real-valued elements) such that

$$A = EDE^\top$$

- Orthogonality of E: write

$$E = [e_1, \cdots, e_p]$$

  then $e_j^\top e_k = 0$ for $j \neq k$ and $\|e_j\| = 1$ for all j.
- Projection onto $\mathcal{C}(e_j)$ is given by $e_j(e_j^\top e_j)^{-1}e_j^\top = e_j e_j^\top$

Eigendecomposition Suppose that A be a symmetric matrix. Let λ_j be the jth diagonal element of D, then we can write

$$A = EDE^\top = \sum_{j=1}^{p} \lambda_j e_j e_j^\top$$

We can know that A is the sum of orthogonal projection operators. e_js are eigenvector and λ_j is the associated eigenvalue. $\mathcal{C}(e_j)$ is eigenspace spaned by e_j.

For simplicity let A be 2×2 matrix.

- Let $D_1 = \text{diag}(\lambda_1, 0)$ and $D_2 = \text{diag}(\lambda_2, 0)$, then

$$D_1 E^\top = \lambda_1 \begin{pmatrix} e_1^\top \\ 0 \end{pmatrix} \text{ and } D_2 E^\top = \lambda_2 \begin{pmatrix} 0 \\ e_2^\top \end{pmatrix}$$

- We can easily show that

$$\begin{pmatrix} e_1 & e_2 \end{pmatrix} \begin{pmatrix} e_1^\top \\ e_2^\top \end{pmatrix} = e_1 e_1^\top + e_2^\top e_2$$

Thus,

$$A = EDE^\top = E(D_1 E^\top + D_2 E^\top) = \lambda_1 e_1 e_1^\top + \lambda_2 e_2^\top e_2$$

Eigendecomposition

This eigendecomposition can be viewed as the decomposition of a linear map:

$$\mathcal{L}_A = \sum_{j=1}^{p} \lambda_j \mathcal{L}_{E_j},$$

where $E_j = e_j e_j^\top$.

Note that

- projection onto $\mathcal{C}(e_j)$ is given by $e_j(e_j^\top e_j)^{-1} e_j^\top = e_j e_j^\top$

Therefore,

$$\mathcal{L}_A(x) = \sum_{j=1}^{p} \lambda_j \mathcal{L}_{E_j}(x),$$

where $\mathcal{L}_{E_j}(x)$ is projection onto the jth eigenspace.

Approximation of Linear map

Let $A^{(k)} = \sum_{j=1}^{k} \lambda_j e_j e_j^\top$ then $A^{(k)}$ approximates A?

$$
\begin{aligned}
EDE^\top \mathbf{x} &= [e_1, \cdots, e_p]\mathsf{diag}(\lambda_1, \cdots, \lambda_p) \begin{pmatrix} e_1^\top \\ \vdots \\ e_p^\top \end{pmatrix} \mathbf{x} \\
&= [e_1, \cdots, e_p]\mathsf{diag}(\lambda_1, \cdots, \lambda_p) \begin{pmatrix} e_1^\top \mathbf{x} \\ \vdots \\ e_p^\top \mathbf{x} \end{pmatrix} \\
&= [e_1, \cdots, e_p] \begin{pmatrix} \lambda_1 e_1^\top \mathbf{x} \\ \vdots \\ \lambda_p e_p^\top \mathbf{x} \end{pmatrix} \\
&= \sum_{j=1}^{p} e_j(\lambda_j e_j^\top \mathbf{x}) = (\sum_{j=1}^{p} \lambda_j e_j e_j^\top)\mathbf{x},
\end{aligned}
$$

Eigendecomposition shows the linear map of a symmetric matrix as the composition of three operations:

$$Ax = EDE^\top x$$

$$\begin{aligned} x \mapsto E^\top x \text{ (rotation)} &\mapsto D(E^\top x) \text{ (scaling)} \\ &\mapsto E(DE^\top x) \text{ (reverse rotation)} \end{aligned}$$

Inverse matrix of positive definite matrix

Let $\mathbf{A}$ be symmetric and nonnegative definite matrix. Then the minimum eigenvalue is positive if and only if $\mathbf{A}$ is positive definite.

pf) Let λ_{min} be the minumum eigenvalue of $\mathbf{A}$. Assume that $\lambda_{min} > 0$. Let $\mathbf{x} = \sum_{j=1} a_j e_j \neq 0$, then

$$\mathbf{x}^\top \mathbf{A}\mathbf{x} = \sum_{j=1}^{p} \lambda_j (e_j^\top \mathbf{x})^2 = \sum_{j=1}^{p} \lambda_j a_j^2 > 0.$$

Assume that $\mathbf{A}$ is pd matrix. WLOG, let λ_p be the minimum eigenvalue of $\mathbf{A}$. Then,

$$e_p^\top \mathbf{A} e_p = \sum_{j=1} \lambda_j (e_j^\top e_p)^2 = \lambda_p > 0.$$

Inverse matrix of positive definite matrix

The inverse matrix of such $\mathbf{A}$ is given by

$$\mathbf{A}^{-1} = \mathbf{E}D^{-1}\mathbf{E}^\top.$$

pf) $\mathbf{E}D^{-1}\mathbf{E}^\top\mathbf{A} = \mathbf{E}D^{-1}\underbrace{\mathbf{E}^\top\mathbf{E}}_{=I}D\mathbf{E}^\top = I$

and $\mathbf{A}\mathbf{E}D^{-1}\mathbf{E}^\top = \mathbf{E}D\underbrace{\mathbf{E}^\top\mathbf{E}}_{=I}D^{-1}\mathbf{E}^\top = I$. By definition of the inverse matrix, we obtain the result.

잠깐!

특별히 pd matrix $A \in \mathbb{R}^{p \times p}$에 대해서

$$\lambda_{\min} = \min_x \frac{x^\top A x}{\|x\|^2}, \quad \lambda_{\max} = \max_x \frac{x^\top A x}{\|x\|^2}$$

라 놓으면 $\lambda_{\max} \geq \lambda_{\min} > 0$ 이며, $\lambda_{\max}$, $\lambda_{\min}$ 각각 A의 maximum eigenvalue, minimum eigenvalue에 해당한다. 한편 pd matrix A에서 Eigenmatrix의 열 $e_1, \cdots e_p$는 다음과 같이 구할 수 있다.

- $e_1 = \text{argmax}_{x \in \mathbb{R}^p} \frac{x^\top A x}{\|x\|^2}$
- $e_2 = \text{argmax}_{x \in \mathbb{R}^p : x \perp e_1} \frac{x^\top A x}{\|x\|^2}$
- $e_3 = \text{argmax}_{x \in \mathbb{R}^p : x \perp \mathcal{C}([e_1, e_2])} \frac{x^\top A x}{\|x\|^2}$
- $\cdots$

Singular Value Decomposition

Let $\mathbf{X} \in \mathbb{R}^{n \times p}$ $(p < n)$ denote a (row-centered) data matrix whose rows are observations and columns are variables. The *singular value decomposition* (SVD) factorises $\mathbf{X}$ as

$$\mathbf{X} = \mathbf{U}\,\mathbf{\Sigma}\,\mathbf{V}^\top,$$

where

- $\mathbf{U} = \left[U_1, \ldots, U_p\right] \in \mathbb{R}^{n \times p}$ contains the left singular vectors $\;(\mathbf{U}^\top \mathbf{U} = \mathbf{I}_p)$;
- $\mathbf{V} = \left[V_1, \ldots, V_p\right] \in \mathbb{R}^{p \times p}$ contains the right singular vectors $\;(\mathbf{V}^\top \mathbf{V} = \mathbf{I}_p)$;
- $\mathbf{\Sigma} = \mathrm{diag}(\sigma_1, \ldots, \sigma_p)$ with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p > 0$;
- $r = \mathrm{rank}(\mathbf{X}) \leq \min\{n, p\}$.

Singular Value Decomposition: Example i

Let

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

be a real matrix with rank $p = 2$. The **reduced singular value decomposition** (SVD) of $\mathbf{A}$ is given by

$$\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top$$

where

- $\mathbf{U} = \begin{bmatrix} -0.5026 & 0.7746 \\ -0.5740 & -0.6325 \\ -0.6464 & 0.0000 \end{bmatrix} \in \mathbb{R}^{3\times 2}$ contains the left singular vectors (orthonormal: $\mathbf{U}^\top\mathbf{U} = \mathbf{I}_2$),

Singular Value Decomposition: Example ii

- $\mathbf{\Sigma} = \begin{bmatrix} 5.1962 & 0 \\ 0 & 1.7321 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ is the diagonal matrix of singular values,
- $\mathbf{V} = \begin{bmatrix} -0.7071 & -0.7071 \\ -0.7071 & 0.7071 \end{bmatrix} \in \mathbb{R}^{2\times 2}$ contains the right singular vectors (orthonormal: $\mathbf{V}^\top \mathbf{V} = \mathbf{I}_2$).

Thus, the matrix $\mathbf{A}$ can be approximately reconstructed as

$$\mathbf{A} \approx \mathbf{U}\mathbf{\Sigma}\mathbf{V}^\top = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix}.$$

Data and representation

$(A)_{ij}$ denotes the entry of A in the row i and the column j. For $i = 1, \cdots, n$,

$$x_i = (X)_{i1} e_1 + \cdots + (X)_{ip} e_p,$$

where $\{e_1, \ldots, e_p\}$ forms the standard basis of $\mathbb{R}^p$.

- Coordinate system: $(e_1, \cdots, e_p)$
- Scaling factor: $(1, \ldots, 1) \in \mathbb{R}^p$
- Representation of x_i w.r.t the $(e_1, \cdots, e_p)$: $((X)_{i1}, \cdots, (X)_{ip}) \in \mathbb{R}^p$.

How to obtain low dimensional representation of x_i effectively?

SVD: Definition & Structure

Full SVD (rank p)

$$\mathbf{X} = \underbrace{\mathbf{U}}_{n\times p}\underbrace{\mathbf{\Sigma}}_{p\times p}\underbrace{\mathbf{V}^\top}_{p\times p} = \sum_{i=1}^{p} \sigma_i\, U_i V_i^\top, \quad \sigma_1 \geq \cdots \geq \sigma_p > 0.$$

$$\mathbf{U}\mathbf{\Sigma} = \begin{bmatrix} U_1 & \cdots U_p \end{bmatrix} \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_p \end{bmatrix} = \begin{bmatrix} \sigma_1 U_1 & \cdots \sigma_p U_p \end{bmatrix}$$

SVD: Definition & Structure

$$\mathbf{X} = \begin{bmatrix} \sigma_1 U_1 & \cdots & \sigma_p U_p \end{bmatrix} \begin{bmatrix} V_1^\top \\ V_2^\top \\ \vdots \\ V_p^\top \end{bmatrix} = \sigma_1 \; U_1 V_1^\top + \cdots + \sigma_p U_p V_p^\top$$

The example below helps understanding the above equation:

$$\underbrace{\begin{bmatrix} u_{11} & u_{21} \\ u_{12} & u_{22} \\ u_{13} & u_{23} \end{bmatrix}}_{\mathbf{U} \in \mathbb{R}^{3\times 2}} \underbrace{\begin{bmatrix} \sigma_1 v_{11} & \sigma_1 v_{21} \\ \sigma_2 v_{12} & \sigma_2 v_{22} \end{bmatrix}}_{\mathbf{\Sigma V}^\top \in \mathbb{R}^{2\times 2}} = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \beta_{11} U_1 + \beta_{21} U_2 & \beta_{12} U_1 + \beta_{22} U_2 \end{bmatrix}$$

$$= [\beta_{11} U_1 \;\; \beta_{12} U_1] + [\beta_{21} U_2 \;\; \beta_{22} U_2]$$

$$= U_1 [\beta_{11} \;\; \beta_{12}] + U_2 [\beta_{21} \;\; \beta_{22}]$$

SVD: Definition & Structure

Our conclusion is that a data matrix $\mathbf{X} \in \mathbb{R}^{n\times p}$ has the following representation,

$$\mathbf{X} = \underbrace{\mathbf{U}}_{n\times p}\underbrace{\mathbf{\Sigma}}_{p\times p}\underbrace{\mathbf{V}^\top}_{p\times p} = \sum_{i=1}^{p} \sigma_i U_i V_i^\top, \quad \sigma_1 \geq \cdots \geq \sigma_p > 0.$$

SVD and representation

Denote the ith row vector of $\mathbf{X}$ by $x_i^\top$. Then,

$$\mathbf{X} = \begin{bmatrix} x_1^\top \\ \vdots \\ x_n^\top \end{bmatrix} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^\top = \begin{bmatrix} U_1 & \cdots & U_p \end{bmatrix} \underbrace{\begin{bmatrix} \sigma_1 V_1^\top \\ \cdots \\ \sigma_p V_p^\top \end{bmatrix}}_{=\boldsymbol{\Sigma}\mathbf{V}^\top}.$$

By taking transpose operator on $\mathbf{X}$

$$\begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} = \begin{bmatrix} \sigma_1 V_1 & \cdots & \sigma_p V_p \end{bmatrix} \mathbf{U}^\top.$$

Thus, $x_1 = \sigma_1 V_1 \times (U^\top)_{11} + \sigma_2 V_2 \times (U^\top)_{21} + \cdots + \sigma_k V_k \times (U^\top)_{p1}$, which implies $(\sigma_1 (U^\top)_{11}, \cdots, \sigma_p (U^\top)$ is a representation of x_1 with respect to $(V_1, \cdots, V_p)$.

SVD and representation

For $i = 1, \cdots, n$,

$$x_i = \sigma_1 (U^\top)_{1i} \times V_1 + \sigma_2 (U^\top)_{2i} \times V_2 + \cdots + \sigma_p (U^\top)_{pi} \times V_p$$

and $(U^\top)_{ji} = U_{ij}$, the following interpretations are derived from SVD.

- New (orthonormal) coordinate system: $(V_1, \cdots, V_p)$
- Scaling factor: $(\sigma_1, \cdots, \sigma_p)$
- Representation of x_i w.r.t the $(V_1, \cdots, V_p)$: $(\sigma_1 U_{i1}, \cdots, \sigma_p U_{ip})$.

Singular Value Decomposition: Example i

$$\mathbf{A} = \begin{bmatrix} 3 & 1 \\ 2 & 2 \\ 1 & 3 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

There is three observation in $\mathbb{R}^2$. $x_1^\top = (3, 1)$, the representation of the ith obs with respect to $(1, 0)^\top, (0, 1)^\top$.

$$\mathbf{U} = \begin{bmatrix} -0.5026 & 0.7746 \\ -0.5740 & -0.6325 \\ -0.6464 & 0.0000 \end{bmatrix} \in \mathbb{R}^{3\times 2}$$

The first row of $\mathbf{U}$, $(-0.5026, 0.7746)$ is the representation of x_1 with respect to $(-0.7071, -0.7071)$ and $(-0.7071, 0.7071)$.

Rank-k Truncated SVD: Definition & Structure

Full SVD (rank p)

$$\mathbf{X} = \sum_{i=1}^{p} \sigma_i U_i V_i^\top, \quad \sigma_1 \geq \cdots \geq \sigma_p > 0.$$

If $\sigma_j \simeq 0$ for all $j > k$, then $\mathbf{X} \simeq \sum_{i=1}^{k} \sigma_i U_i V_i^\top$. That is, x_i is represented based on the basis $\{V_1, \cdots, V_k\}$ of a k-dimensional subspace and $(U_{i1}, \cdots, U_{ik}) \in \mathbb{R}^k$ is the rank-reduced representation of x_i.

Rank-2 Truncated SVD: Visualization

Let $\mathbf{X}_k = \sum_{i=1}^k \sigma_i U_i V_i^\top$. and the denote ith row vector of $\mathbf{X}_k$ by $\tilde{x}_i^\top$.

- Axis: V_1 (horizontal) and V_2 (vertical)
- Interpretations of the Axis:
  $V_1 = (v_{11}, \cdots, v_{1p})^\top = \sum_{j=1}^p v_{1j} e_j$ and $V_2 = (v_{21}, \cdots, v_{2p})^\top = \sum_{j=1}^p v_{1j} e_j$, where e_js are the standard basis. V_j is explained by the covariates's names of the data and the associated coefficients $(v_{j1}, \cdots, v_{jp})$.
  (ex) Suppose that $V_1 = (0.7101, -0.7101, 0, \cdots, 0)$, X_1: GDP, X_2 : interest rate, then V_1 is the weighted sum of GDP and interest rate with the weight $(0.7101, -0.7101)$.
- Poisiton of $\tilde{x}_i^\top$: $(\sigma_1 U_{i1}, \sigma_2 U_{i2})$.