Vector Optimization

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Dual Cone and Generalized inequality

Definition 1 (Cone)

A set $K \subseteq \mathbb{R}^n$ is a *cone* if

$$x \in K,\ \theta \geq 0 \quad \Rightarrow \quad \theta x \in K.$$

Definition 2 (Proper Cone)

A cone $K \subseteq \mathbb{R}^n$ is called a *proper cone* if it satisfies:

1. **Convex:** $x, y \in K \ \Rightarrow\ \theta x + (1-\theta) y \in K,\ \forall \theta \in [0,1]$.
2. **Closed:** K is a closed set.
3. **Solid:** $\mathrm{int}(K) \neq \emptyset$.
4. **Pointed:** $x \in K$ and $-x \in K \ \Rightarrow\ x = 0$.

Pointed condition 은 cone 내에서 방향성있는 비교를 위해서 도입한 조건임.

Cone 을 이용하여 $\mathbb{R}^n$ 위에서 partial ordering 을 정의

- $x \preceq_K y \Leftrightarrow y - x \in K$
- $x \prec_K y \Leftrightarrow y - x \in \text{int}(K)$

partial ordering 으로 잘 정의되는가 확인할 수 있음

- $x \preceq_K x$ for all x 는 $x - x = 0 \in K$ 를 확인
- $x \preceq_K y$ and $y \preceq_K x$ 는 $x = y$ 는 $y - x \in K$ and $x - y \in K \to x = y$ 임을 확인
- $x \preceq_K y$ and $y \preceq_K z \to x \preceq_K z$는 $y - x \in K$ and $z - y \in K \to z - x \in K$ 임을 확인

즉, $\preceq_K$ 가 partial ordering 임을 보일 수 있음

Examples 3

Let $K = \{c \in \mathbb{R}^b | c_1 + c_2 t + \cdots + c_n t^{n-1} \geq 0$ for $t \in [0,1]\}$. K is cone of polynomials of degree $n-1$ that are nonnegagive on $[0,1]$. It can be showns that K is proper.

For $c, d \in \mathbb{R}^n$, $c \preceq_K d$ if and only if

$$c_1 + c_2 t + \cdots + c_n t^{n-1} \leq d_1 + d_2 t + \cdots + d_n t^{n-1}$$

for all $x \in [0,1]$.

Definition 4 (Minimum Element)

Let $S \subseteq \mathbb{R}^n$ and $\preceq_K$ be a generalized inequality induced by a cone K. An element $x \in S$ is a *minimum element* of S if

$$x \preceq_K y, \quad \forall y \in S.$$

If it exists, the minimum element is unique.

만약, $x \in S$가 minimum element of S w.r.t a cone K라 하자. 그러면 이는 $S \subseteq x + K$ 와 동치.

Definition 5 (Minimal Element)

An element $x \in S$ is a *minimal element* of S if there is no $y \in S$ such that

$$y \preceq_K x \quad \text{and} \quad y \neq x$$

In general, a set can have many minimal elements.

만약 x가 S의 minimal element 라고 하면 $x - K \ \cap S = \{x\}$와 동치다.

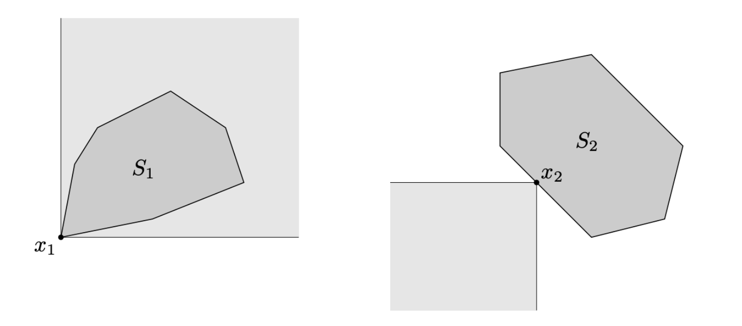

Figure 1: K is $\mathbb{R}^2_+$: x_1 is the mininum element of S_1 and x_2 is a minimal element of S_2.

Definition 6 (Dual Cone)

Given a cone $K \subseteq \mathbb{R}^n$, the *dual cone* K^* is defined as

$$K^* = \{v \in \mathbb{R}^n \mid v^T x \geq 0,\ \forall x \in K\}.$$

- 의미1: dual cone의 기하학적 의미는 K 안의 모든 벡터와 90도 이하의 각을 이루는 벡터들의 집합.
- 의미2: $v : \mathbb{R}^n \mapsto \mathbb{R}$ 인 linear functional 로 보았을때 ordering K를 보존하는 linear function 모임.

$v \in K^*$ can be regarded as a map $v : \mathbb{R}^n \mapsto \mathbb{R}$. Following below shows that v preserves the order w.r.t K on the real valued space.

Let $x \preceq_K y$ and let $v \in K^*$.

- $y - x \in K$.
- $v^\top (y - x) \geq 0$ for all $v \in K^*$ by definition. That is,

$$v^\top x \leq v^\top y$$

Illustration of Dual Cone

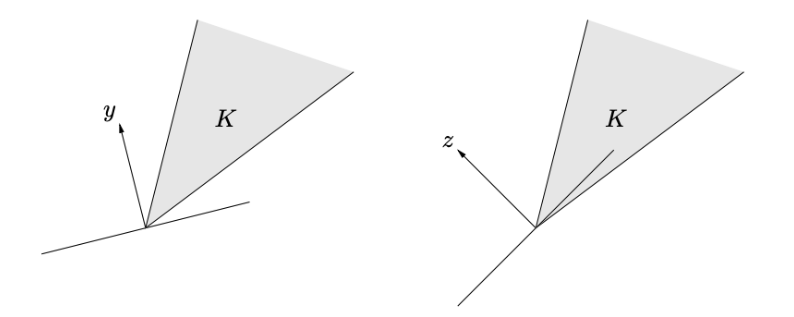

Figure 2.22 *Left.* The halfspace with inward normal y contains the cone K, so $y \in K^*$. *Right.* The halfspace with inward normal z does not contain K, so $z \notin K^*$.

Figure 2: Caption

Example 7 (Nonnegative orthant is self-dual)

Consider the cone $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n\}$. Its dual cone is defined as

$$(\mathbb{R}^n_+)^* = \{y \in \mathbb{R}^n \mid y^T x \geq 0,\ \forall x \in \mathbb{R}^n_+\} = \mathbb{R}^n_+$$

Proof:

- If $y \in \mathbb{R}^n_+$, then for any $x \in \mathbb{R}^n_+$, $y^T x = \sum_{i=1}^n y_i x_i \geq 0 \quad \Rightarrow \quad y \in (\mathbb{R}^n_+)^*$.
- If $y \notin \mathbb{R}^n_+$, then there exists j such that $y_j < 0$. Take $x = e_j \in \mathbb{R}^n_+$, the j-th standard basis vector. Then

$$y^T x = y_j < 0,$$

contradicting the definition of the dual cone.

Hence,

$$(\mathbb{R}^n_+)^* = \mathbb{R}^n_+.$$

Definition 8 (Norm Cone and Dual Norm Cone)

Given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the *norm cone* is defined as

$$C = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} \mid \|x\| \le t\}.$$

The dual cone is

$$C^* = \{(y,\alpha) \in \mathbb{R}^n \times \mathbb{R} \mid \|y\|_* \le \alpha,\ \alpha \ge 0\},$$

where $\|\cdot\|_*$ denotes the dual norm. Note that given a norm $\|\cdot\|$ on $\mathbb{R}^n$, the *dual norm* $\|\cdot\|_*$ is defined as $\|y\|_* = \sup_{\|x\|\le 1} y^T x$.

(See the Example 2.25 in p52)

Let $K \subseteq \mathbb{R}^n$ be a cone. The dual cone K^* has the following properties:

1. **Closed convex cone:** K^* is always a closed convex cone.
2. **Inclusion reversal:** If $K_1 \subseteq K_2$, then

$$K_2^* \subseteq K_1^*.$$

3. If K has nonempty interior, the K^* is pointed.
4. If K is closed and pointed, then K^* has nonempty interior.
5. If K is convex and closed, $K^{**} = K$
6. If K is a proper cone, then so is its dual K^* and moreover $K^{**} = K$.

Definition 9 (Generalized Inequality)

Let $K \subseteq \mathbb{R}^n$ be a proper cone. For $x, y \in \mathbb{R}^n$, we define

$$x \preceq_K y \iff y - x \in K,$$

and the strict inequality as

$$x \prec_K y \iff y - x \in \operatorname{int}(K).$$

Proposition 1 (Dual Generalized Inequality)

Let $K \subseteq \mathbb{R}^n$ *be a proper cone with dual cone*

$$K^* = \{z \in \mathbb{R}^n \mid z^T x \geq 0,\ \forall x \in K\}.$$

Then for any $x, y \in \mathbb{R}^n$*,*

$$x \preceq_K y \iff z^T x \leq z^T y \quad \forall z \in K^*,$$

$$x \prec_K y \iff z^T x < z^T y \quad \forall z \in K^* \setminus \{0\}.$$

Generalized inequality는 dual cone을 통해 실수에 대한 부등식으로써 동일하게 표현할 수 있다는 뜻임. (K^* 내의 모든 linear function 에 대해서 값의 순서를 유지한다는 뜻)

Theorem 10 (Alternatives for linear strict generalized ineqaulities)

Let $K \subseteq \mathbb{R}^m$ be a proper cone. $Ax \prec_K b$ is infeasible where $x \in \mathbb{R}^m$ if and only if there exists λ such that

$$\lambda \neq 0, \quad \lambda \succeq_{K^*} 0, \quad A^\top \lambda = 0, \quad \lambda^\top b \leq 0.$$

proof) ($\Rightarrow$) Define two convex sets

$$S_1 = \{\, b - Ax \mid x \in \mathbb{R}^m \,\}, \qquad S_2 = \operatorname{int} K.$$

The assumption that the system $Ax \prec_K b$ is infeasible means $S_1 \cap S_2 = \emptyset$. Since S_1 is an affine set and S_2 is an open convex set, the separating hyperplane theorem implies that there exist $\lambda \neq 0 \in \mathbb{R}^m$ and $\mu \in \mathbb{R}$ such that

$$\lambda^\top (b - Ax) \le \mu \quad \text{for all } x, \tag{1}$$

$$\lambda^\top y \ge \mu \quad \text{for all } y \in \operatorname{int} K. \tag{2}$$

(1) is impossible unless $A^\top \lambda = 0$. Thus $\lambda^\top b \le \mu$. (2) is only possible when $\lambda \in K^*$ and $\mu \le 0$. (If $\mu = \mu_0 > 0$ is set, then $\lambda^\top y \ge \mu_0 / t$ for $t > 0$ since $ty \in$ int$K$.)

Combining with $\lambda^\top b \le \mu$ and $\mu \le 0$ gives $\lambda^\top b \le 0$. In summary, the infeasibility implies that

$$\lambda \neq 0, \quad \lambda \succeq_{K^*} 0, \quad A^\top \lambda = 0, \quad \lambda^\top b \le 0.$$

($\Leftarrow$) Suppose that ..

Proposition 2 (Characterization of Minimum Element)

Let K be a proper cone. An element $x^\star \in S$ is a minimum element of S if and only if

$$\lambda^T x^\star \leq \lambda^T x, \quad \forall x \in S,\ \forall \lambda \in K^*,$$

where K^* is the dual cone of K.

Proposition 3 (Minimum element and supporting hyperplanes)

An element $x^\star \in S$ is a minimum element of S if and only if for every $\lambda \in$ int K^*, the hyperplane

$$H_\lambda = \{x \in \mathbb{R}^n \mid \lambda^T x = \lambda^T x^\star\}$$

is a supporting hyperplane of S at $x^\star$.

(Recall) If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$, then the hyperplane $\{x : a^\top x = a^\top x_0\}$ is called a supporting hyperplane to C.

Illustration of Dual Cone

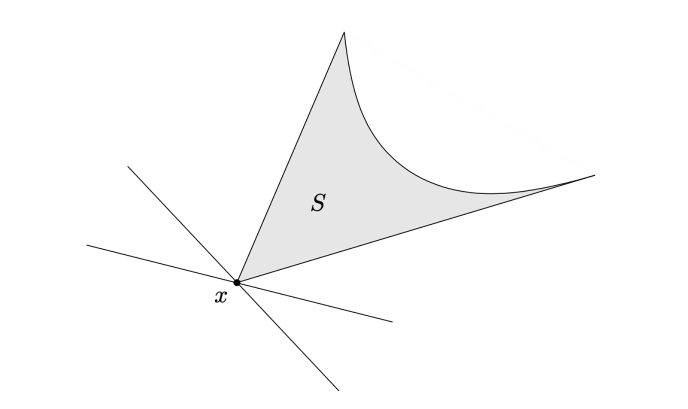

Figure 3: Caption

Proposition 4 (Dual Characterization of Minimal Elements)

Let $S \subseteq \mathbb{R}^n$ and $\preceq_K$ be a generalized inequality induced by a proper cone K with dual cone K^*. An element $x^* \in S$ is a minimal element of S if there exists some $\lambda \in$ int K^* such that

$$\lambda^T x^* < \lambda^T x, \quad \forall x \in S \setminus \{x^*\}.$$

(proof) Suppose that x^* is not minimal of S. There exists $z \in S$ such that $z \preceq_K x^*$ and $z \neq x^*$. $\lambda^\top z \leq \lambda^\top x^* < \lambda^\top x \quad$ for all $x \in S \setminus \{x^*\}$, which is contradiction.

Note that the converse is in general false. When S is convex, the converse is true for $\lambda \succeq_K 0$.

Illustration of Dual Cone

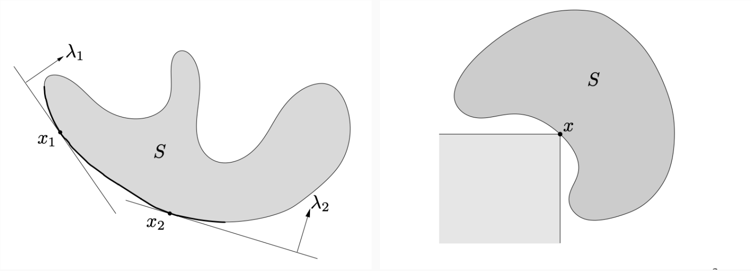

Figure 4: Left: x_1 and x_2 are minimal of S; Right: x is a minimal of S but there is such a λ in Proposition 4

Definition 11 (Pareto Optimal Production Point)

Let $K = \mathbb{R}^m_+ = \{u \in \mathbb{R}^m \mid u_i \geq 0,\ i = 1, \ldots, m\}$ and $S \subseteq \mathbb{R}^m_+$ be the production possibility set. A point $x^\star \in S$ is *Pareto optimal* (i.e., a K-minimal element of S) if

$$\nexists\, x \in S \text{ such that } x \preceq_K x^\star \text{ and } x \neq x^\star.$$

Equivalently, there is no feasible point that (componentwise) produces at least as much of every good and strictly more of at least one good.

The *Pareto optimal production frontier* of S is the set of all Pareto optimal points:

$$\mathcal{P}(S) \;=\; \left\{\, x^\star \in S \;\middle|\; \nexists\, x \in S \text{ with } x \preceq_K x^\star,\ x \neq x^\star \,\right\}.$$

Convexity with respect to generalized inequality

Definition 12 (K-nondecreasing function)

Let $K \subseteq \mathbb{R}^n$ be a proper cone. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be K-*nondecreasing* if

$$x \preceq_K y \quad \Rightarrow \quad f(x) \leq f(y).$$

- K-nondecreasing 함수는 cone K가 정의하는 순서에 대해 단조증가하는 함수.
- $K = \mathbb{R}^n_+$ 면 우리가 아는 좌표별 증가 함수와 같다.

Proposition 5 (Gradient condition for monotonicity)

Let $K \subseteq \mathbb{R}^n$ be a proper cone and $f : \mathbb{R}^n \to \mathbb{R}$ be differentiable. Then f is K-nondecreasing if and only if

$$\nabla f(x) \succeq_{K^*} 0 \quad \forall x \in \mathbb{R}^n$$

That is, $\nabla f(x) \in K^*, \quad \forall x \in \mathbb{R}^n$, where K^* is the dual cone of K.

Definition 13 (K-convex function)

Let $K \subseteq \mathbb{R}^m$ be a proper cone. A function $F : \mathbb{R}^n \to \mathbb{R}^m$ is said to be K-*convex* if

$$F(\theta x + (1-\theta)y) \ \preceq_K \ \theta F(x) + (1-\theta)F(y), \quad \forall x, y \in \mathbb{R}^n,\ \theta \in [0,1].$$

Equivalently,

$$\theta F(x) + (1-\theta)F(y) - F(\theta x + (1-\theta)y) \in K.$$

Proposition 6 (Scalarization of K-convexity)

Let $K \subseteq \mathbb{R}^m$ be a closed convex cone with dual K^*, and let $F : \mathbb{R}^n \to \mathbb{R}^m$.

1. *If F is K-convex, then for every $w \in K^*$ the scalar function $\phi_w(x) = w^T F(x)$ is convex.*
2. *Conversely, if ϕ_w is convex for every $w \in K^*$, then F is K-convex.*

($\Rightarrow$) Assume F is K-convex. Then for all x, y and $\lambda \in [0,1]$ we have

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in K.$$

Let $w \in K^*$. By the definition of the dual cone, $\langle w, z\rangle \geq 0 \quad \forall\, z \in K$. Applying $\langle w, \cdot\rangle$ to the above K-inclusion yields

$$\lambda\, w^\top F(x) + (1-\lambda)\, w^\top F(y) - w^\top F(\lambda x + (1-\lambda)y) \geq 0.$$

Thus

$$w^\top F(\lambda x + (1-\lambda)y) \leq \lambda\, w^\top F(x) + (1-\lambda)\, w^\top F(y),$$

which shows that $w^\top F$ is convex.

($\Leftarrow$) Assume that $w^\top F$ is convex for every $w \in K^*$. Then for all x, y, $\lambda \in [0,1]$, and $w \in K^*$, $w^\top F(\lambda x + (1-\lambda)y) \le \lambda\, w^\top F(x) + (1-\lambda)\, w^\top F(y)$. Rearranging,

$$\langle w,\, \lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y)\rangle \ge 0 \quad \forall\, w \in K^*.$$

Thus,

$$\lambda F(x) + (1-\lambda)F(y) - F(\lambda x + (1-\lambda)y) \in (K^*)^*.$$

Since K is a closed convex cone, it is bidual, i.e. $(K^*)^* = K$. Hence,

$$F(\lambda x + (1-\lambda)y) \preceq_K \lambda F(x) + (1-\lambda)F(y),$$

showing that F is K-convex.

Proposition 7 (Gradient condition for K-convexity)

Let $K \subseteq \mathbb{R}^m$ be a proper cone and $F : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable. Then F is K-convex if and only if

$$F(y) \succeq_K F(x) + \nabla F(x)(y - x), \quad \forall x, y \in \mathbb{R}^n,$$

where $\nabla F(x)$ denotes the Jacobian of F at x.

Theorem 14 (Composition rule for K-convex functions)

Let $K \subseteq \mathbb{R}^m$ be a proper cone. Suppose $F : \mathbb{R}^n \to \mathbb{R}^m$ is K-convex and $h : \mathbb{R}^m \to \mathbb{R}$ is convex and K-nondecreasing. Then the composition

$$\phi(x) = h(F(x))$$

is a convex function on $\mathbb{R}^n$.

Vector Optimization

Problem 15 (Vector Optimization with Constraints)

$$\begin{aligned} \min_{x \in \mathbb{R}^n} \; & F(x) = (f_1(x), f_2(x), \ldots, f_m(x)) \text{ \textit{w.r.t} } K \\ \textit{s.t.} \; & g_i(x) \leq 0, \quad i = 1, \ldots, p, \\ & h_j(x) = 0, \quad j = 1, \ldots, q. \end{aligned}$$

(That is, by criterion $F(x) \preceq_K F(y)$)

Definition 16 (Pareto optimal point)

A feasible point $x^* \in \mathcal{X}$ is called *Pareto optimal* if there is no $x \in \mathcal{X}$ such that

$$F(x) \preceq_K F(x^*) \quad \text{and} \quad F(x) \neq F(x^*).$$

Definition 17 (Achievable objective values)

Let $\mathcal{X}_{\text{feas}}$ denote the feasible set of a vector optimization problem. The set of *achievable objective values* is

$$\mathcal{F} = \{\, F(x) \in \mathbb{R}^m \;\mid\; x \in \mathcal{X}_{\text{feas}} \,\}.$$

Proposition 8 (Optimality condition via achievable set)

Let $\mathcal{O} = \{F(x) \mid x \in \mathcal{X}_{\text{feas}}\}$ be the set of achievable objective values, and let K be a proper cone. Then a feasible point x^* is Pareto optimal if and only if

$$\mathcal{O} \subset F(x^*) + K.$$

Proposition 9 (Characterization of Pareto optimal points)

Let $F : \mathbb{R}^n \to \mathbb{R}^m$ *be the vector objective,* $\mathcal{X}_{\text{feas}}$ *the feasible set, and*

$$\mathcal{O} = \{F(x) \mid x \in \mathcal{X}_{\text{feas}}\}$$

the set of achievable objective values. Let $K \subseteq \mathbb{R}^m$ *be a proper cone.*

A feasible point $x^* \in \mathcal{X}_{\text{feas}}$ *is Pareto optimal if and only if*

$$\mathcal{O} \subset F(x^*) + K,$$

where

$$F(x^*) + K = \{\, F(x^*) + k \mid k \in K \,\}.$$

Remark 1

The condition $\mathcal{O} \subset F(x^*) + K$ means that every achievable objective value is greater than or equal to $F(x^*)$ with respect to the generalized inequality $\preceq_K$. Thus $F(x^*)$ is a minimal element of the set $\mathcal{O}$, and therefore x^* is a Pareto optimal solution.

Proposition 10 (Scalarization $\Rightarrow$ Pareto optimality)

Let $X \subset \mathbb{R}^n$ be convex and $F : X \to \mathbb{R}^m$. Let $K \subset \mathbb{R}^m$ be a proper cone with nonempty interior, and let K^* denote its dual cone. Suppose that $w \in \operatorname{int} K^*$ and $x^\star \in X$ satisfies

$$w^\top F(x^\star) \ \leq \ w^\top F(x) \quad \forall x \in X.$$

Then $x^\star$ is Pareto optimal (i.e. K-efficient).

($\Rightarrow$) Assume for contradiction that $x^\star$ is not Pareto optimal. Then there exists x such that $F(x) \preceq_K F(x^\star)$ and $F(x) \neq F(x^\star)$. By the definition of the K-order, this means $F(x^\star) - F(x) \in K$. Since $w \in K^*$ and $F(x^\star) - F(x) \in K$, we have

$$\langle w, F(x^\star) - F(x) \rangle > 0.$$

Equivalently,

$$w^\top F(x^\star) > w^\top F(x),$$

which contradicts the optimality of $x^\star$ for the scalarized problem $\min_{x \in X} w^\top F(x)$.

Hence such an x cannot exist, and $x^\star$ must be Pareto optimal.

Proposition 11 (Pareto optimality $\Rightarrow$ Scalarization in the K-convex case)

Let $X \subset \mathbb{R}^n$ be a nonempty convex set and $F : X \to \mathbb{R}^m$ be K-convex, where $K \subset \mathbb{R}^m$ is a closed, convex, proper cone. Assume that the upper image

$$C := F(X) + K := \{\, F(x) + k \mid x \in X,\ k \in K \,\}$$

is closed and convex. Let $x^\star \in X$ be a Pareto optimal point. Then there exists a nonzero $w \in K^*$ such that

$$x^\star \in \arg\min_{x \in X} w^\top F(x).$$

Nondominated Sorting Genetic Algorithm-II

Nondominated Sorting Genetic Algorithm-II (maximization prob) (Link)

- Initialization: Generate initial population $P_0 = \{x_1, \dots, x_N\}$ and set $t = 0$.
- Combine parent and offspring: (for $t > 0$)
  If $t > 0$, let $R_t \leftarrow P_t \cup Q_t$; otherwise $R_0 \leftarrow P_0$.
- Fast Nondominated Sorting:
  Apply nondominated sorting to R_t to obtain fronts $F_1, F_2, \dots,$ where
  $\mathcal{F}_1 = \{x \in R_t : \nexists\, y \in R_t,\ F(y) \prec_K F(x)\}$, and, inductively for $k \geq 2$,

$$\mathcal{F}_k = \left\{ x \in R_t \setminus \bigcup_{i=1}^{k-1} \mathcal{F}_i : \nexists\, y \in R_t \setminus \bigcup_{i=1}^{k-1} \mathcal{F}_i,\ F(y) \prec_K F(x) \right\}.$$

  For each $x \in R_t$, set $\mathrm{rank}(x) = k$ if $x \in \mathcal{F}_k$.

- Crowding Distance Assignment:
  For each front $\mathcal{F}_k$, compute the crowding distance $d(x)$ for all $x \in \mathcal{F}_k$ by

$$d(x) = \sum_{j=1}^{m} \frac{f_j(x^{(i+1)}) - f_j(x^{(i-1)})}{\max_{y \in \mathcal{F}_k} f_j(y) - \min_{y \in \mathcal{F}_k} f_j(y)},$$

  where $x = x^{(i)}$ is the i-th solution in $\mathcal{F}_k$ sorted by objective f_j, and boundary solutions are assigned $d(x) = \infty$.
- Environmental Selection (Form P_{t+1}):
  $P_{t+1} \leftarrow \emptyset, \quad k \leftarrow 1$
  While $(|P_{t+1}| + |F_k| \le N)$:
  $\quad P_{t+1} \leftarrow P_{t+1} \cup F_k$
  $\quad k \leftarrow k + 1$
  Let $L = N - |P_{t+1}|$ be the remaining slots.
  From F_k, choose L individuals with the largest crowding distances and add them to P_{t+1}.

- Mating Selection (Binary Tournament):
  Use the binary tournament selection (Algorithm 1) with comparison rule $x >_{\mathsf{nsga}} y$ based on $(\mathsf{rank}(x), d(x))$ to obtain a mating pool M_t of size N from P_{t+1}.
- Variation (Genetic Operators):
  Apply crossover and mutation to M_t to generate the offspring population Q_t of size N:

$$Q_t = \mathsf{Variation}(M_t) = \mathsf{Mutation}(\mathsf{Crossover}(M_t)).$$

Algorithm 1: Binary Tournament Selection in NSGA-II

Input: Population P of size N with assigned $(\mathrm{rank}(x), d(x))$ for all $x \in P$

Output: Mating pool M of size N

Comparison rule (NSGA-II):

$$x \succ_{\mathsf{NSGA}} y \iff \begin{cases} \mathrm{rank}(x) < \mathrm{rank}(y), \\ \text{or } \big(\mathrm{rank}(x) = \mathrm{rank}(y) \text{ and } d(x) > d(y)\big). \end{cases}$$

$M \leftarrow \emptyset$;

while $|M| < N$ **do**

  Select two individuals x, y independently and uniformly at random from P ;

  If $x \succ_{\mathsf{NSGA}} y$ then $M \leftarrow M \cup \{x\}$;

  else $M \leftarrow M \cup \{y\}$;

end

return M

Algorithm 2: Variation operator in NSGA-II

Input: Mating pool M of size N

Output: Offspring population Q of size N

$Q \leftarrow \emptyset$;

for $i = 1$ **to** $N/2$ **do**

  select two parents p_1, p_2 from M;

  $(c_1, c_2) \leftarrow$ SBX_Crossover(p_1, p_2);

  $c_1 \leftarrow$ PolynomialMutate(c_1);

  $c_2 \leftarrow$ PolynomialMutate(c_2);

  $Q \leftarrow Q \cup \{c_1, c_2\}$;

end

return Q

Simulated Binary Crossover: SBX_Crossover(p_1, p_2) returns c_1 and c_2

$$u \sim \mathcal{U}(0,1), \qquad \beta_q = \begin{cases} (2u)^{\frac{1}{\eta_c+1}}, & u \le \frac{1}{2}, \\ \left(\dfrac{1}{2(1-u)}\right)^{\frac{1}{\eta_c+1}}, & u > \frac{1}{2}, \end{cases}$$

$$c_{1j} = \frac{1}{2}\big[(1+\beta_q)p_{1j} + (1-\beta_q)p_{2j}\big], \qquad c_{2j} = \frac{1}{2}\big[(1-\beta_q)p_{1j} + (1+\beta_q)p_{2j}\big].$$

($\eta_c > 0$ is hyperparameter.)

Polynomial Mutation: PolynomialMutate(c) returns

$$\delta_1 = \frac{x_j - L_j}{U_j - L_j}, \qquad \delta_2 = \frac{U_j - x_j}{U_j - L_j}, \qquad u \sim \mathcal{U}(0,1),$$

$$\Delta_j = \begin{cases} \left(2u + (1-2u)(1-\delta_1)^{\eta_m+1}\right)^{\frac{1}{\eta_m+1}} - 1, & u < \frac{1}{2}, \\ \\ 1 - \left(2(1-u) + 2(u-\frac{1}{2})(1-\delta_2)^{\eta_m+1}\right)^{\frac{1}{\eta_m+1}}, & u \geq \frac{1}{2}, \end{cases}$$

$$x'_j = x_j + \Delta_j(U_j - L_j).$$

U_j, L_j are upper and lower bound associated a searching range.