Convex Optimization Problem II

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Fall, 2026

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Things to know

- Convex set and examples
- Visualization of a halfspace
- Convex preserving map

Convex set A subset $\mathcal{C}$ of $\mathbb{R}^n$ is convex if and only if for all $x_1, x_2 \in \mathcal{C}$ and all $\theta \in [0, 1]$,

$$\theta x_1 + (1 - \theta) x_2 \in \mathcal{C}$$

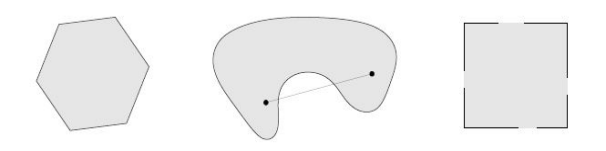


Figure 1: One convex set and two nonconvex sets

Cone: A set $K \subset \mathbb{R}^n$ is a cone if $x \in K \rightarrow \alpha x \in K$ for any $\alpha \geq 0 \in \mathbb{R}$.

Convex cone C is convex and cone, which means that

$$\theta_1 x_1 + \theta_2 x_2 \in C$$

for any $x_1, x_2 \in C$ and $\theta_1, \theta_2 \geq 0$.

Convex cone is convex.

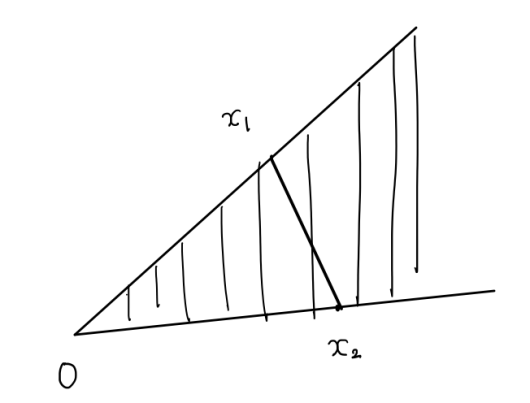


Figure 2: Convex cone

Norm cone

$$\mathcal{C} = \{(x, t) \in \mathbb{R}^n \times \mathbb{R}_+ : \|x\|_2 \leq t\}$$

Note that

$$C = \{(x, t) \in \mathbb{R}^{n+1} : \begin{pmatrix} x \\ t \end{pmatrix}^\top \begin{pmatrix} I & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} \leq 0, t \geq 0\}$$

Norm cone is convex.

Example (norm cone)

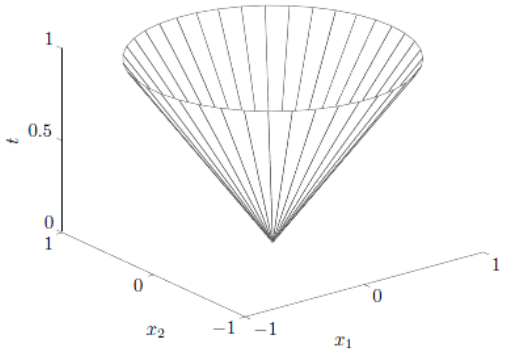


Figure 2.10 Boundary of second-order cone in $\mathbf{R}^3$, $\{(x_1, x_2, t) \mid (x_1^2+x_2^2)^{1/2} \leq t\}$.

Conic Combination

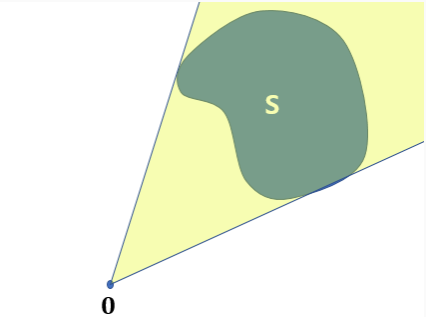
Definition

Let $S \subseteq \mathbb{R}^n$. A vector $x \in \mathbb{R}^n$ is called a **conic combination** of vectors $x_1, x_2, \ldots, x_k \in S$ if

$$x = \sum_{i=1}^{k} \theta_i x_i, \quad \text{where } \theta_i \geq 0 \ \ \forall i.$$

Conic hull: the conic hull of a set S is the set of all conic combinations of the points in S,

$$\text{Cone}(S) = \bigcup_{n=1}^{\infty} \left\{ \sum_{i=1}^{n} \alpha_i x_i : \alpha_i \geq 0, x_i \in S \right\}.$$



(Conic hull) The conic hull of a set C is the smallest convex cone containing C.

(proof)

- Conic hull of a set C is convex and cone.
- If C' is convex cone containing C then the conic hull of C is contained by C'.
- Thus, the proof is complete.

Ray: $\{x_0 + \theta\nu : \theta \geq 0\}$

A ray is convex.

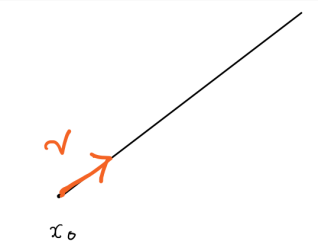


Figure 3: Visualization of Ray

Hyperplane and Halfsapce

- Hyperplane: $\{x \in \mathbb{R}^n : a^\top x = b\}$
- Halfspace: $\{x \in \mathbb{R}^n : a^\top x \leq b\}$

Both hyperplane and halfspace are convex. Hyperplain divides the space into the two halfspace. Each halfspace is the SOLUTION SET OF THE INEQUALITY.

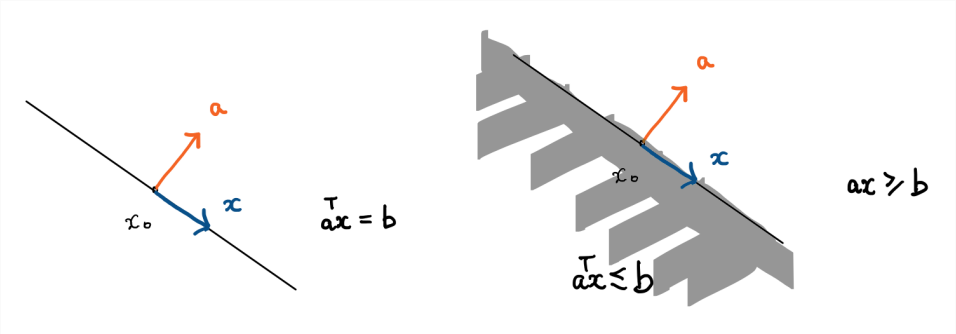
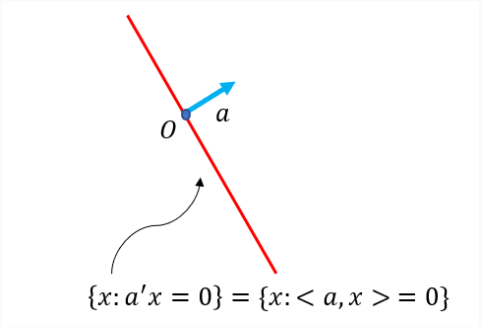


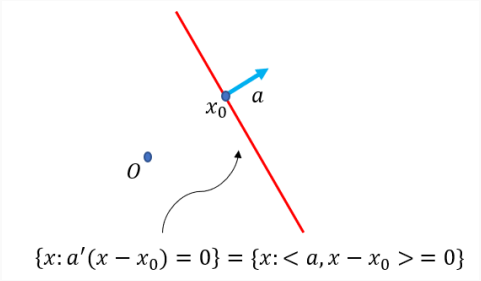
Figure 4: Hyperplane and Halfspace

Visualization of half space

We know that $C = \{x : a^\top x = 0\}$ is illustrated by line on a plane and the vector can be denoted by normal vector at the origin of the line. Let $x = a$ then $a^\top x > 0$ and we know that the $x = a \in \{x : a^\top x \geq 0\}$. That is, the half space containing the denoted a becomes $\{x : a^\top x \geq 0\}$.

Let $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^n$. Fix an a and consider a set $\{x : a^\top x = 0\}$. The figure below shows how to display $\{x \in R^n : a^\top x = b\}$ on graph. First consider a set $\{x \in R^n : a^\top x = 0\}$.





Unless $a \neq 0$, we can choose $x_0 \in \mathbb{R}^n$ satisfying $a^\top x_0 = b$. Similarly, we can denote a as a perpendicular vector on the line.

($\{x : a^\top x \geq b\}$ 영역의 시각화 방법)

- $\{x : a^\top x = b\}$ 을 나타내는 직선을 그린다.
- 직선의 위쪽의 한 점 x_0를 선택한 후 $a^\top x_0$를 계산한다.
- $a^\top x_0 \geq 0$ 가 참이면 해당 영역을 칠하고, 거짓이면 반대쪽 영역을 칠한다.

Practice 1

Consider $a, x \in \mathbb{R}^2$

- Let $a = (2, 1)^\top$ and draw the point a in $\mathbb{R}^2$.
- Draw the line, the set of points (x_1, x_2) satisfying $2x_1 + x_2 = 0$.
- Check that the line is the set $C = \{x : a^\top x = 0\}$.
- Make a shade on the region $2x_1 + x_2 \geq 0$ and check that the region contains a.
- Finally check that a is orthogonal to all elements in C.

Practice 2

Consider $A = \mathbb{R}^{m\times 2}$ and $x \in \mathbb{R}^2$. Denote $a_j^\top \in \mathbb{R}^2$ for $j = 1, \cdots, m$ be the jth row vector of A. We will illustrate the region $\{x : Ax \preceq 0\}$.

- Note that $\{x : Ax \preceq 0\} = \cap_{j=1}^m \{x : a_j^\top x \leq 0\}$.
- Draw the region, the set of points (x_1, x_2) satisfying $a_j^\top x \leq 0$ for each j.
- Find the intersection of the regions.

Examples (l_2-ball)

$\mathcal{C} = \{\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n : x_1^2 + \cdots + x_n^2 \leq 1\}$

Euclidean norm: $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + \cdots + x_n^2}$.

$$\mathbf{x} \in \mathcal{C} \Leftrightarrow \|\mathbf{x}\|_2 \leq 1$$

For $\mathbf{x}, \mathbf{x}' \in \mathcal{C}$,

$$\begin{aligned} \|\theta\mathbf{x} + (1-\theta)\mathbf{x}'\|_2 &\leq \|\theta\mathbf{x}\|_2 + \|(1-\theta)\mathbf{x}'\|_2 \\ &= \theta\|\mathbf{x}\|_2 + (1-\theta)\|\mathbf{x}'\|_2 \leq 1 \end{aligned}$$

$\theta\mathbf{x} + (1-\theta)\mathbf{x}' \in \mathcal{C}$.

The first inequality holds because of triangular inequality.

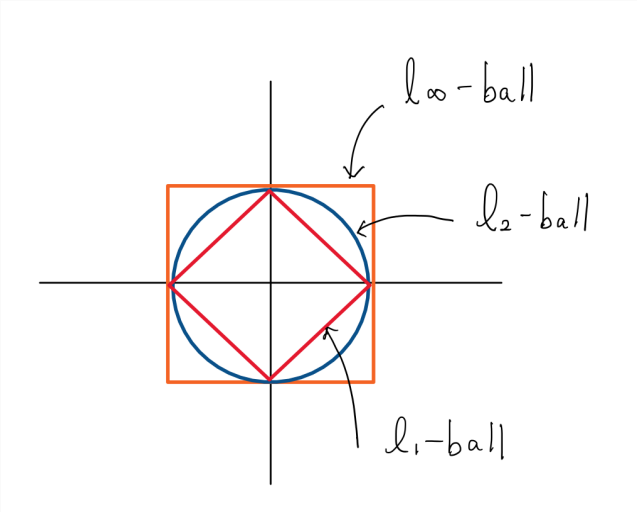


Figure 5: l_p-ball

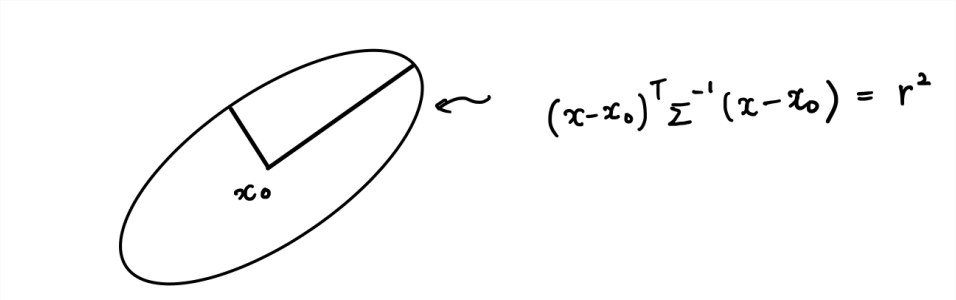


Figure 6: Ellipsoid

Examples (polyhedron)

Let A be $n \times p$ matrix, $b \in \mathbb{R}^p$, $\mathbf{C}$ be $m \times p$ matrix, and $d \in \mathbb{R}^m$.

$C = \{x \in \mathbb{R}^n : Ax \leq b, Cx = d\}$ is convex.

위 예는 Convex set의 정의를 이용해서 확인할 수 있는 예다.

(Proof) Let $x, x' \in C$ For $\theta \in [0, 1]$

$$A(\theta x + (1-\theta)x') = \theta Ax + (1-\theta)Ax' \leq \theta b + (1-\theta)b = b,$$

and

$$C(\theta x + (1-\theta)x') = \theta Cx + (1-\theta)Cx' = d$$

So, $\theta x + (1-\theta)x' \in C$

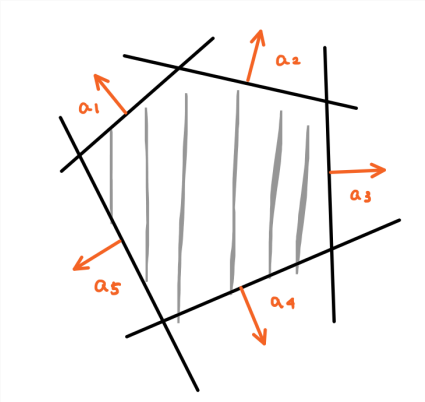


Figure 7: Visualization of Polyhedron

Examples (Semipositive definite matrix S_+^n)

$$S^n = \{X \in \mathbb{R}^{n \times n} : X = X^\top\}$$

$$S_+^n = \{X \in S^n | X \succeq 0\}$$

$$S_{++}^n = \{X \in S^n | X \succ 0\}$$

All sets are convex.

Supporting hyperplane

Let $C \subset \mathbb{R}^n$ and x_0 be a point on the boundary of C. If $a \neq 0$ satisfies $a^\top x \leq a^\top x_0$ for all $x \in C$ then, the hyperplane $\{x : a^\top x = a^\top x_0\}$ is called a supporting hyperplane to C at x_0.

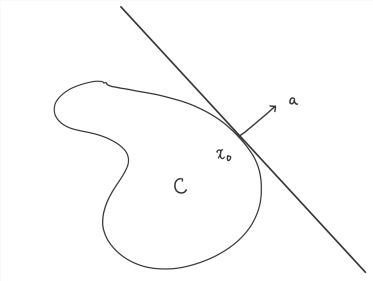


Figure 8: Supporting Hyperplane

Convex preserving operation

- If C_1, C_2 are convex sets, then $C_1 \cap C_2$ is also convex set.
- If C is convex set, then $C + z = \{x + z : x \in C\}$ is convex.
- Let C be convex subset of $\mathbb{R}^m$ and $f : x \in C \mapsto Ax + b \in \mathbb{R}^n$ where $A \in \mathbb{R}^{n \times m}$ and $b \in \mathbb{R}^n$. Then $\{y : y = Ax + b, x \in C\}$ is convex.
- Perspective function*
- Linear fractional functions*

Perspective function

$f : \mathbb{R}^n \times \mathbb{R}_{++} \mapsto f(x, z) \in \mathbb{R}^n$, where $f(x, z) = x/z$.

The perspective function preserves the convexity of a set. If $C \subset \text{dom}(f)$ is convex, then $f(C)$ is convex. In addition, an inverse image of f is convex.

Linear fractional function

Let $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. $f : \mathbb{R}^n \mapsto f(x) \in \mathbb{R}^m$, where

$$f(x) = \frac{Ax + b}{c^\top x + d}, \quad \text{dom}(f) = \{x : c^\top x + d > 0\}.$$

Linear fractional function Let $g : \mathbb{R}^n \mapsto \mathbb{R}^{m+1}$ is affine, i.e.,

$$g(x) = \begin{pmatrix} A \\ c^\top \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix},$$

where $A \in \mathbb{R}^{m \times n}$ and $b \in \mathbb{R}^m$ and $c \in \mathbb{R}^n$ and $d \in \mathbb{R}$. Let P be a perspective function, then

$$f(x) = P \circ g.$$

This representation of f helps to prove that the linear-fractional function is convex preserving.

Example: probability model

- $w = (w_1, \ldots, w_n) \in \mathbb{R}^n$, $w_j \geq 0$ and suppose that $w \in C$ (convex)
- probability model:

$$q_i = \frac{w_i}{\sum_{j=1}^n w_j}, \quad i = 1, \ldots, n.$$

Then,

$$Q = \{q : q = \frac{w}{1^\top w}, w \in C\}$$

is convex.

HW

- Prove that the linear fractional function is convexity preserving map.
- Solve 2.1-2.10, 2.16-2.19

사영공간

- 사영공간(projective space) $\mathbb{P}^n$은 $\mathbb{R}^{n+1}$에서 원점을 지나는 모든 직선의 집합. $\mathbb{P}^n$에서의 한 원소는 $\mathbb{R}^{n+1}$에서 원점을 지나는 직선 하나에 대응됨. 원점이 아닌 $x = (x_0, x_1, \ldots, x_n) \neq 0$에 0이 아닌 스칼라 λ에 대해

$$L_x = \{\lambda x \mid \lambda \in \mathbb{R}\}$$

  는 x로 식별되는 $\mathbb{R}^{n+1}$ 위의 직선 하나.

- $\mathbb{P}^n$에서는 직선 L_x 전체를 묶어 하나의 점으로 취급함. $[1:2:3]$과 $[2:4:6] = [-1:-2:-3]$은 $\mathbb{R}^3$에서는 같은 직선 위의 점들이며, 사영공간 $\mathbb{P}^2$ 내에서 모두 하나의 점에 대응됨.

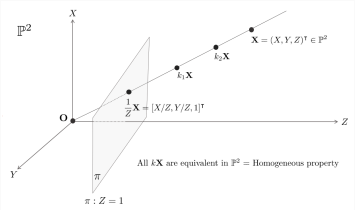


Figure 9: Illustration of $\mathbb{P}^2$

정의

Equivalent class에 의한 정의는 $\mathbb{P}^n \equiv \left(\mathbb{R}^{n+1} \backslash \{0\}\right) / \sim$ where $\mathbf{u} \sim \mathbf{v} \iff \exists k \neq 0 : \mathbf{u} = k\mathbf{v}$.

- 3차원에서 한 점을 $\mathbf{X} = [X, Y, Z]^\top$로 쓰면, 모든 $k\mathbf{X}$가 $\mathbb{P}^2$에서 같은 원소.

동차좌표(homogeneous coordinates)

- $\mathbb{P}^2$의 한 점은 $\mathbb{R}^3$위 원점을 지나는 한 직선에 대응되므로 그 식별자를 동차좌표 $[X : Y : Z]$로 표기.
- 사영공간의 정의에 의해 $[X : Y : Z] = [kX : kY : kZ]$ $(k \neq 0)$.
- $Z \neq 0$이면 **아핀 차트** $Z = 1$로 정규화하여 표현가능: $[X : Y : Z] \mapsto \left[\frac{X}{Z}, \frac{Y}{Z}, 1\right]^\top$.
- 이때, 대응하는 유클리드 좌표는 $\left(\frac{X}{Z}, \frac{Y}{Z}\right) \in \mathbb{R}^2$이며 2차원상의 3차원의 직선하나 혹은 사영공간의 한 점을 식별할 수 있음.

- 아핀 차트(π)위의 직선 $L = \{x : (a, b, c)^\top x = 0\}$ 을 생각하자.
- 여기서 직선 L을 $l = (a, b, c)$ 라고 표기하자. 그러면 $x \in \pi$에 대해서 항상 $x^\top l = 0$ 을 만족함.
- π 위에서 두 직선 L과 L'이 있고 이 두 직선이 π 위의 한 점 x에서 만난다고 하자.
- 교점 x는 $x^\top l = x^\top l' = 0$를 만족하므로 x는 열공간 $\mathcal{C}([l, l'])$에 직교하는 벡터와 같음. (참고: 이는 외적 $l \times l'$으로 계산할 수 있음)

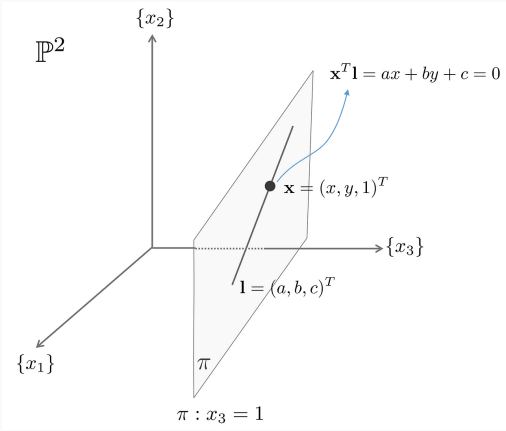


Figure 10: Caption

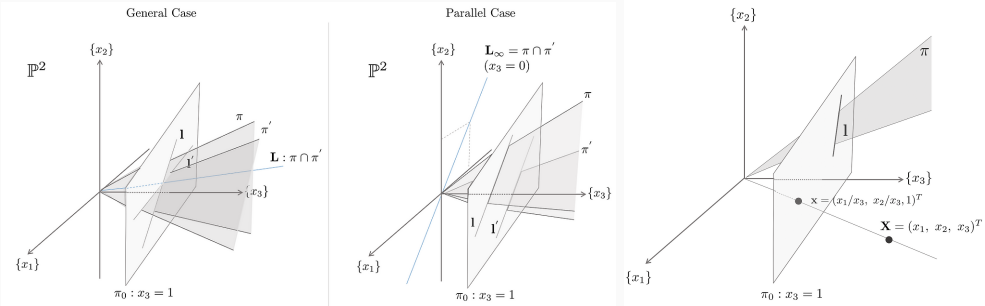


Figure 11: A model for the projective plane

Duality (점-직선의 대칭성)

- 사영공간에서는 **점과 직선이 대칭(duality)**을 가짐.
- 예시:
  - 직선 ℓ 위의 한 점 $\mathbf{x}$는 $\mathbf{x}^\top \ell = 0$ 또는 $\ell^\top \mathbf{x} = 0$ 두 가지 방식으로 표현 가능.
  - 두 직선 ℓ, ℓ'이 교차하는 한 점: $\mathbf{x} = \ell \times \ell'$
  - 두 점 $\mathbf{x}, \mathbf{x}'$를 통과하는 직선: $\ell = \mathbf{x} \times \mathbf{x}'$
- 즉, 점과 직선의 위치를 바꾸어도 동일한 공식을 사용 가능.
- 따라서, **두 점을 지나는 직선의 공식**은 **두 직선이 교차하는 점의 공식**과 대칭을 이룸.

평면 위의 일반적인 이차 곡선(conic)은 다음과 같이 주어진다:

$$ax^2 + bxy + cy^2 + dx + ey + f = 0.$$

여기서 항들의 차수는 서로 다르다:

- ax^2, bxy, cy^2 는 2차항,
- dx, ey 는 1차항,
- f 는 상수항(0차항).

이를 **동차식(homogeneous polynomial)**으로 바꾸기 위해 새로운 변수 z를 도입하여 모든 항을 **2차**로 맞춘다.

$$\begin{aligned} dx &\longrightarrow dxz, \\ ey &\longrightarrow eyz, \\ f &\longrightarrow fz^2. \end{aligned}$$

따라서 동차화(homogenization)된 식은

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

이 되며, 이는 **2차 동차식**이다.

conic 방정식 $F(x, y, z) = 0$이 있다고 하자. 예를 들어,

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

는 **2차 동차식**이다. 따라서 (x, y, z)가 해라면, 임의의 스칼라 $k \neq 0$에 대해

$$F(kx, ky, kz) = k^2 F(x, y, z) = 0$$

이므로 (kx, ky, kz)도 역시 해가 된다. 즉, 원점에서 출발하는 한 직선 위의 모든 점들이 conic 위의 동일한 사영점(projective point)에 해당한다.

따라서 conic은 사영공간에서 $\{(x : y : z) \in \mathbb{P}^2 \mid F(x, y, z) = 0\}$ 로 정의된다.

The conic function can be written by

$$\begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

where

$$A = \begin{bmatrix} a & b/2 & d/2 \\ b/2 & c & e/2 \\ d/2 & e/2 & f \end{bmatrix}.$$

일반적인 conic의 방정식은

$$ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = 0$$

이를 벡터 $\mathbf{c} = (a\ b\ c\ d\ e\ f)^T \in \mathbb{R}^6$로 쓰면,

$$(x_i^2 \ \ x_i y_i \ \ y_i^2 \ \ x_i \ \ y_i \ \ 1)\mathbf{c} = 0$$

로 나타낼 수 있다.

다섯 개의 점 $(x_i, y_i),\ i = 1, \ldots, 5$를 대입하면

$$A\mathbf{c} = 0$$

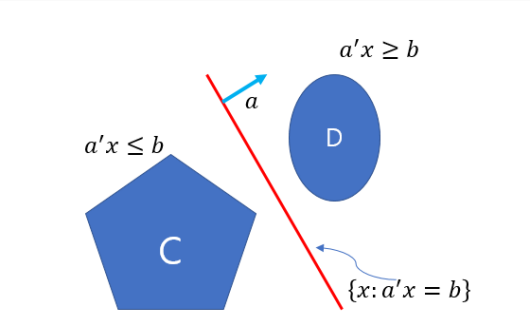
여기서 행렬 A는

$$A = \begin{bmatrix} x_1^2 & x_1 y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2 y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3 y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4 y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5 y_5 & y_5^2 & x_5 & y_5 & 1 \end{bmatrix}$$

행렬의 A null space 벡터가 유일한 의 해가 되고 conic을 유일하게 결정

Separating hyperplane theorem

Let C and D be convex sets in $\mathbb{R}^n$ with $C \cap D = \phi$. Then, there exists $a \in \mathbb{R}^n$, $a \neq 0$ and $b \in \mathbb{R}$, such that $a^\top x \leq b$ for all $x \in C$ and $a^\top x \geq b$ for all $x \in D$.



Strict separating hyperplane theorem

In general, a strict separating hyperplane does not hold, even when C and D are closed.

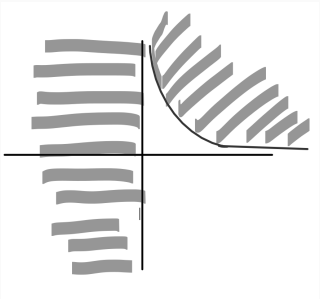


Figure 12: Not strict separating convex sets

Appendix

Strict separating hyperplane theorem

Let C be a closed convex set and $D = \{x_0\}$ with $x_0 \notin C$. Then, there exists $a \neq 0$ and b such that $a^\top x < b$ for all $x \in C$ and $a^\top x > b$ for $x \in D$.

This result implies that a convex is represented by all intersections of hyperplanes containing the convex set.

Appendix

Converse separating hyperplane theorem

See p50.